JOURNAL OF COMPUTER AND SYSTEM SCIENCES: 3, 335–360 (1969)

A Decomposition Method for Structured Linear and Nonlinear Programs*

M. D. GRIGORIADIS† AND K. RITTER‡

Computer Sciences Department, University of Wisconsin, Madison, Wisconsin

Received August 23, 1968

ABSTRACT

A decomposition method for nonlinear programming problems with structured linear constraints is described. The structure of the constraint matrix is assumed to be block diagonal with a few coupling constraints or variables, or both. The method is further specialized for linear objective functions. An algorithm for performing post optimality analysis—ranging and parametric programming—for such structured linear programs is included. Some computational experience and results for the linear case are presented.

1. INTRODUCTION

In practice, large nonlinear programming problems with linear constraints, as well as large linear programs, almost always exhibit some structure in their constraint matrix. The most common of these structures is the block diagonal structure with a few coupling constraints or variables or both. To date, various methods for the solution of such large problems with *either* coupling constraints *or* coupling variables (both linear and nonlinear) and linear, quadratic, separable or general nonlinear objective functions have been developed (see, e.g., [*1–4*, 9, *11*]). In [*16*, *17*], Rosen describes partition methods which use the special block diagonal structure of the constraints to reduce the given problem by elimination of variables.

A common assumption in all decomposition or partitioning methods known to the authors is that the constraint matrix represents a "weakly coupled" system: The number of coupling constraints or coupling variables, or both, is assumed to be much smaller than the corresponding dimension of the problem. Violation of this rather

* This work was sponsored in part by the Mathematics Research Center, United States Army, Madison, Wisconsin, under Contract No.: DA-31-124-ARO-D-462 and in part by the Department of Computer Sciences, University of Wisconsin NSF Research Grant GP6070.

† New York Scientific Center, IBM Corporation, New York City.

‡ Mathematics Research Center, University of Wisconsin, Madison, Wisconsin.

571/3/4-1

qualitative criterion reportedly has led to poor convergence and other computational irregularities.

The block diagonal structure with a small number of coupling constraints and variables frequently arises in dynamic formulations of multiplant, multi-commodity production scheduling and distribution models in various industries. This type of a linear model can be converted into the familiar block diagonal structure with only coupling constraints (or only coupling variables) but this conversion results in a drastic increase in the number of such coupling constraints (or variables). Thus, the most desirable property of this inherently weakly coupled system is sacrificed.

This paper describes a decomposition or partitioning method [*13*] which uses the special structure of the constraints to reduce the given problem through elimination of variables. If may be readily applied to problems having a block diagonal structure with coupling constraints or variables, or both. The objective function is assumed to be nonlinear, differentiable and concave in all variables. Dual feasibility is maintained throughout the optimization procedure.

The method is further specialized to the case of a linear objective function, first treated by Ritter [*12*] as a generalization of Rosen's Primal Partition Programming [*17*]. In addition, an algorithm for performing postoptimality studies for the linear case [*14*] is offered. This uses the computational tools developed for the linear version of the proposed decomposition algorithm.

In the next section, the nonlinear problem is defined and the basic idea of the method, to be detailed in Section 3, is summarized. In Section 4, the simplifications arising from the linearity of the objective function are discussed. The postoptimality algorithm is given in Section 5. The validity of the proposed algorithm is demonstrated in Section 6. In the final section some computational aspects and our experience with this algorithm are presented.

2. The Nonlinear Problem

We consider the following problem:

Maximize

$$F(y, x_1, \ldots, x_k) \tag{2.1}$$

subject to the linear constraints

$$\sum_{j=1}^{k} A_j x_j + D_0 y = b_0 \tag{2.2}$$

$$B_j x_j + D_j y = b_j \qquad (j = 1, \ldots, k) \tag{2.3}$$

$$y \geqslant 0;\ x_j \geqslant 0 \qquad (j = 1, \ldots, k), \tag{2.4}$$

where $F(y, x_1, \ldots, x_k)$ is a differentiable and concave function, A_j $(j = 1, \ldots, k)$ is an (m_0, n_j)-matrix, B_j $(j = 1, \ldots, k)$ is an (m_j, n_j)-matrix, D_j $(j = 0, 1, \ldots, k)$ is an (m_j, n_0)-matrix, while x_j and c_j are n_j-vectors, y is a n_0-vector and b_j $(j = 0, 1, \ldots, k)$ is an m_j-vector. This problem will be referred to as the Primal Problem (**P**). The corresponding dual is:

Minimize

$$F(y, x_1, \ldots, x_k) - u_0' \left(\sum_{j=1}^{k} A_j x_j + D_0 y - b_0 \right)$$

$$- \sum_{j=1}^{k} u_j'(B_j x_j + D_j y - b_j) + w_0' y + \sum_{j=1}^{k} w_j' x_j, \tag{2.5}$$

subject to the constraints

$$\sum_{j=1}^{k} D_j' u_j + D_0' u_0 - w_0 - \nabla_y F(y, x_1, \ldots, x_k) = 0, \tag{2.6}$$

$$B_j' u_j + A_j' u_0 - w_j - \nabla_{x_j} F(y, x_1, \ldots, x_k) = 0 \qquad (j = 1, \ldots, k), \tag{2.7}$$

$$w_j \geqslant 0 \qquad (j = 0, 1, \ldots, k), \tag{2.8}$$

where the u_j $(j = 0, 1, \ldots, k)$ and w_j $(j = 0, 1, \ldots, k)$ represent the dual variables or Lagrange multipliers and are m_j, and n_j-vectors respectively; $\nabla_y F$ is a n_0-vector corresponding to the portion of ∇F which consists of the partial derivatives of F with respect to the components of y only, and $\nabla_{x_j} F$ are n_j-vectors corresponding to the portions of ∇F which consist of partial derivatives of F with respect to the components of x_j only.

A more convenient form of the dual problem, which will be referred to as **D**, may be obtained by eliminating the variables w_j $(j = 0, 1, \ldots, k)$ from (2.5)–(2.7) and using (2.8). This is given by:

Minimize

$$F(y, x_1, \ldots, x_k) + \sum_{j=0}^{k} u_j' b_j - (y', x_1', \ldots, x_k'). \nabla F(y, x_1, \ldots, x_k), \tag{2.9}$$

subject to

$$\sum_{j=1}^{k} D_j' u_j + D_0' u_0 - \nabla_y F(y, x_1, \ldots, x_k) \geqslant 0, \tag{2.10}$$

$$B_j' u_j + A_j' u_0 - \nabla_{x_j} F(y, x_1, \ldots, x_k) \geqslant 0 \qquad (j = 1, \ldots, k), \tag{2.11}$$

The decomposition method described in this paper is mainly based on the following observation. If **P** has an optimal solution, then the variables in this solution have nonnegative values. Since we have a total of $n = \sum_{j=0}^{k} n_j$ variables, we would expect to have, at most, as many active constraints in **P**. However, the $m = \sum_{j=0}^{k} m_j$ equality constraints (2.2)–(2.3) are always active. Therefore, provided that (2.2)–(2.3) are linearly independent,[1] at most $(n - m)$ of the nonnegativity constraints (2.4) are active. The remaining m nonnegativity restrictions, which are inactive, may be canceled with no effect on the optimal solution of **P**.

A further simplification may be effected by using the special structure of the constraints (2.3) to eliminate at least $(m - m_0)$ of the variables which are not restricted in sign. This elimination procedure reduces the maximization problem **P** to a concave programming problem with at most $s = n - (m - m_0)$ variables, all of which are restricted to be nonnegative, and m_0 linear equality constraints. This problem will be referred to as the Modified Primal Problem (**M**), and may be regarded as analogous to the "master problem" in Dantzig–Wolfe decomposition [*1*] or the "Problem II" in Rosen's Primal Partition Programming [*17*].

Clearly, if the set of nonnegative restrictions (2.4) active in the optimal solution to **P** were known in advance, then the solution of **M** would provide the optimal solution to **P**. Generally, however, it is unlikely that one might predict the optimal basic variable set or equivalently the nonnegativity restrictions which would be active in the optimal solution to **P**. To circumvent this difficulty, we begin by ignoring the nonnegativity restrictions for an arbitrary set S_1 of at least $(m - m_0)$ variables chosen among the x_j. In this case, the optimal solution to **M** need not be feasible for **P** since some of the eliminated variables may take on negative values. If it is feasible, however, then it is also an optimal solution to **P** (Theorem 1).

If some variables have negative values, we determine a new set S_2 of at least $(m - m_0)$ variables and repeat the procedure. It can be shown (Lemmas 1 and 2) that corresponding to the sequence of optimal solutions to the modified primal problems (**M**), there is a sequence of solutions to **D** which give non-increasing values of the dual objective function. From this fact it follows (Theorem 2) that after a finite number of steps, we obtain a modified maximization problem (**M**) which has the same optimal solution as **P**.

Since the appearance of [*11–13, 7*], the name "relaxation methods" has been offered by Geoffrion [*6*] to describe this general class of techniques.

3. The Algorithm

We assume that each of the matrices B_j contains a nonsingular square matrix of order m_j. This is no loss of generality since if B_j does not contain such a matrix, we can add suitable unit vectors and artificial variables having sufficiently large negative

[1] The case of linearly dependent constraints (2.2)–(2.3) may cause a larger number of nonnegativity restrictions to be active which may in turn result in a larger number of constraints in **M** (See Section 3, Case 3).

entries in the objective function. Then, provided that the original problem (**P**) has a feasible solution, the optimal solution to this enlarged problem is identical to that for **P**.

Let B_{j1} be an m_j-order nonsingular square submatrix of B_j. We denote the matrix formed by the remaining columns by B_{j2} and partition A_j, x_j and c_j accordingly into A_{j1}, A_{j2}, x_{j1}, x_{j2}, and c_{j1}, c_{j2}, respectively. Then, the constraints (2.3) can be written as:

$$x_{j1} = B_{j1}^{-1}b_j - B_{j1}^{-1}B_{j2}x_{j2} - B_{j1}^{-1}D_j y. \tag{3.1}$$

Substituting (3.1) into (2.1)–(2.2) we can eliminate the vectors x_{j1} ($j = 1,..., k$) and obtain the "Modified Primal Problem" (**M**) as:

Maximize

$$G(y, x_{12}, x_{22},..., x_{k2}), \tag{3.2}$$

subject to the linear constraints

$$\sum_{j=1}^{k} M_j x_{j2} + M_0 y = b, \qquad x_{j2} \geqslant 0, \qquad y \geqslant 0, \tag{3.3}$$

where the function $G(y, x_{12}, x_{22},..., x_{k2})$ is concave and differentiable since it is obtained from the function $F(y, (x_{11}, x_{12}),..., (x_{k1}, x_{k2}))$ by the linear transformation (3.1); and

$$\begin{aligned} b &= b_0 - \sum_{j=1}^{k} A_{j1}B_{j1}^{-1}b_j, \\ M_0 &= D_0 - \sum_{j=1}^{k} A_{j1}B_{j1}^{-1}D_j, \\ M_j &= A_{j2} - A_{j1}B_{j1}^{-1}B_{j2}. \end{aligned} \tag{3.4}$$

If **M** has no feasible solution, then the original problem **P** has no feasible solution since it contains all constraints (3.3) of **M**. In the following, we assume that **P** has a feasible solution and **M** attains an optimal solution for a finite point $(y, x_{12},..., x_{k2})$. (If not, the precautionary procedure outlined in Section 6, Remark 2, may be used).

We note that **M** is a concave maximization problem considerably smaller than the original problem **P**, with at most s variables and m_0 linear equality constraints (in addition to the nonnegativity restrictions). Efficient and computationally successful methods for the solution of nonlinear programming problems subject to linear constraints developed by Rosen [*15*], Frank and Wolfe [*5*], and others, may be used. The solution of **M**, theoretically, may not be a finite procedure.

Let (y^*, x_{j2}^*) $(j = 1,..., k)$ be an optimal solution to **M**. Substituting this solution into (3.1) we obtain

$$x_{j1}^* = B_{j1}^{-1}b_j - B_{j1}^{-1}B_{j2}x_{j2}^* - B_{j2}^{-1}D_jy^*. \tag{3.5}$$

Now we apply the following optimality criterion (Theorem 1):

If $x_{j1}^* \geqslant 0$ $(j = 1,..., k)$, then $(y^*, x_{j1}^*, x_{j2}^*)$ $(j = 1,..., k)$ is an optimal solution to the original problem **P**.

Suppose x_{j1}^* $(j = 1,..., k_1 \leqslant k)$ has at least one negative component. We construct a new problem **M**, of the form (3.2)–(3.3), in such a way that in the resulting solution $(y^{**}, x_{j1}^{**}, x_{j2}^{**})$ one of the components of x_j $(j = 1,..., k_1)$ which was negative in $(y^*, x_{j1}^*, x_{j2}^*)$ is forced to be nonnegative. This procedure will now be outlined for a general cycle.

(A) Let $(x_{j1}^*)_1, (x_{j1}^*)_2,..., (x_{j1}^*)_l$ be the negative components of x_{j1}^*. Denote the first l rows of the matrix $B_{j1}^{-1}B_{j2}$ by $g_{j1}',..., g_{jl}'$. Furthermore, suppose that x_{j2}^* has q positive components, say the first q components. Then, for each $j \leqslant k_1$, consider the following two cases:

(I) At least one of the components $(g_{ji})_\nu$ $(i = 1,..., l;\ \nu = 1,..., q)$ is nonzero.

(II) $(g_{ji})_\nu = 0$ for $i = 1,..., l;\ \nu = 1,..., q$.[2]

In Case I, let $(g_{ji})_\nu \neq 0$. Denote the νth column of B_{j2} by $h_{j\nu}$. If the ith column of B_{j1} is replaced by $h_{j\nu}$, the new matrix B_{j1}^* is nonsingular since $(B_{j1}^{-1}h_{j\nu})_i = (g_{ji})_\nu \neq 0$ implies that the columns of B_{j1}^* are linearly independent. Thus, replace B_{j1} by B_{j1}^* for any j for which x_{j1}^* has negative components *and* for which Case I holds. Then, the procedure (3.4) which leads to the construction of **M** is applied using the new matrices B_{j1}^{*-1}. It should be noted that those M_j for which $x_{j1}^* \geqslant 0$, are not altered and need not be recomputed.

In Case II, let $(x_{ji}^*)_\nu < 0$. Denote the νth row of $B_{j1}^{-1}B_{j2}$ and $B_{j1}^{-1}D_j$ by $g_{j\nu}'$ and $e_{j\nu}'$, respectively, and the νth component of $B_{j1}^{-1}b_j$ by $\beta_{j\nu}$. Then, add the condition:

$$-(x_{j1})_\nu = g_{j\nu}'x_{j2} + e_{j\nu}'y - \beta_{j\nu} \leqslant 0, \tag{3.6}$$

to the constraints (3.3) of **M** after all changes dictated by Case I have been implemented.

Finally, the new **M** problem is solved resulting in (y^{**}, x_{j2}^{**}) as its optimal solution. The corresponding x_{j1}^{**} $(j = 1,..., k)$ is obtained by inserting this solution into (3.1).

By Theorem 1, $(y^{**}, x_{j1}^{**}, x_{j2}^{**})$ is an optimal solution to **P** if all components of $x_{j1}^{**} \geqslant 0$ $(j = 1,..., k)$.

[2] If $q = 0$, proceed as if Case II holds.

(B) If at least one of the vectors x_{j1}^{**}, has negative components, the "additional constraints" of the form (3.6) are treated as follows:

Case 1. If $g'_{j\nu}x_{j2}^{**} + e'_{j\nu}y^{**} - \beta_{j\nu} < 0$ then this constraint is canceled.

Case 2. if $g'_{j\nu}x_{j2}^{**} + e'_{j\nu}y^{**} - \beta_{j\nu} = 0$ and $(g_{j\nu})_\mu(x_{j2}^{**})_\mu \neq 0$, then the constraint is canceled and the νth column of B_{j1} is replaced by the μth column $h_{j\mu}$ of B_{j2}. The resulting matrix B_{j1}^{**} is nonsingular since $(B_{j1}^{-1}h_{j\mu})_\nu = (g_{j\nu})_\mu \neq 0$ implies that the columns of B_{j1}^{**} are linearly independent.[3]

Case 3. If $g'_{j\nu}x_{j2}^{**} + e'_{j\nu}y^{**} - \beta_{j\nu} = 0$ and $(g_{j\nu})_\mu(x_{j2}^{**})_\mu = 0$ for all μ, then this constraint is left unaltered in **M**. Since in this case

$$g'_{j\nu}x_{j2}^{**} + e'_{j\nu}y^{**} = e'_{j\nu}y^{**} = \beta_{j\nu},$$

M may contain, except for degenerate cases, at most n_0 constraints of the form (3.6) at the conclusion of any cycle.[4] The presence of linear dependence among the rows of the original constraint matrix (2.2)–(2.3) may cause a slight increase in the number of constraints of **M**.

The modification of the "additional constraints" outlined above, completes a decomposition "cycle." We let $x_{j1}^{*} = x_{j1}^{**}$ and start the next cycle at A.

Since in each cycle at most k "additional constraints" are appended to **M**, it follows from Case 3, that, disregarding degeneracy, **M** may contain at most $(n_0 + k)$ "additional constraints" at any cycle.

By Theorem 2, an optimal solution to **P** is obtained after a finite number of cycles.

Remark. The above procedure yields the optimal solution to **P** after a finite number of decomposition cycles, even when only one of the variables negative in the tth cycle is forced to be nonnegative at the $(t + 1)$th cycle. Consequently, it would suffice to append at most one "additional constraint" of the form (3.6) at each cycle. Then, the number of additional constraints involved in any single cycle would reduce to at most $(n_0 + 1)$.

[3] Since $(x_{j1}^{**})_\nu = 0$, this procedure changes only the partitioning of x_j into (x_{j1}, x_{j2}) but not the actual value of x_j^{**}. In the new partitioning $(x_{j1}^{*})_\nu$ belongs to the variables which are restricted to nonnegative values.

[4] This is easily shown by considering the basis M^B of the current **M**. In the case of non-degeneracy, $(x_{j2}^{**})^B > 0$, which implies that $(g'_{j\nu})^B = 0$ for all "additional constraints" ν remaining in **M** after Cases 1 and 2 have been applied. Since the vectors $(g'_{j\nu}, e'_{j\nu})^B$, being part of M^B, must be linearly independent, there can be at most n_0 such vectors. In the degenerate case, some components of $(x_{j2}^{**})^B$ may be zero and thus the corresponding "additional constraints" have $(g'_{j\nu})^B \neq 0$, resulting in a larger number of possible constraints in **M**.

4. The Linear Case—Simplifications

The linear case is characterized by a linear objective function, i.e.

$$F(y, x_1, \ldots, x_k) = c_0'y + \sum_{j=1}^{k} c_j'x_j ,$$

resulting in the Linear Primal problem (**LP**) and leading to the following formulation of the dual problem (**LD**) corresponding to (2.9)–(2.11):

Minimize

$$\sum_{j=0}^{k} b_j'u_j , \tag{4.1}$$

subject to

$$\sum_{j=1}^{k} D_j'u_j + D_0'u_0 \geqslant c_0 , \tag{4.2}$$

$$B_j'u_j + A_j'u_0 \geqslant c_j \qquad (j = 1, \ldots, k). \tag{4.3}$$

The obvious, and most important, simplification resulting from our assumption of a linear objective function is the linearity of **M**, i.e.:

Maximize

$$\alpha + \sum_{j=1}^{k} d_j'x_{j2} + d_0'y, \tag{4.4}$$

subject to

$$\sum_{j=1}^{k} M_j x_{j2} + M_0 y = b, \tag{4.5}$$

$$y \geqslant 0, \qquad x_{j2} \geqslant 0 \qquad (j = 1, \ldots, k) \tag{4.6}$$

where M_0, M_j $(j = 1, \ldots, k)$ and b are given by (3.4) and

$$\alpha = \sum_{j=1}^{k} c_{j1}' B_{j1}^{-1} b_j , \tag{4.7}$$

$$d_0 = c_0 - \sum_{j=1}^{k} (B_{j1}^{-1} D_j)' \, c_{j1} \, ; \qquad d_j = c_{j2} - (B_{j1}^{-1} B_{j2})' \, c_{j1} \, . \tag{4.8}$$

We note that the above linear version of **M**, which will be referred to as **LM**, is an ordinary linear programming problem with m_0 equality constraints and s variables. It may be solved using any of the commercially available linear programming codes.

Although the maximum number of constraints in **LM** may differ from one cycle to the next, the last remark in Section 3 suggests that a constant size of at least $(n_0 + 1)$, and not more than $(n_0 + k)$, rows may be selected in advance and used for all cycles. This will facilitate the use of an existing linear programming code for solving **LM**.

5. Postoptimality Analysis—The Linear Case

We consider the following parametric form of **LP**:
Maximize

$$\sum_{j=1}^{k} (c_j + \lambda f_j)' x_j + (c_0 + \lambda f_0)' y,$$

subject to the constraints

$$\sum_{j=1}^{k} A_j x_j + D_0 y = b_0 + \lambda e_0 ,$$

$$B_j x_j + D_j y = b_j + \lambda e_j \qquad (j = 1,..., k),$$

$$y \geqslant 0; \qquad x_j \geqslant 0 \qquad (j = 1,..., k),$$

where f_j and e_j $(j = 0, 1,..., k)$ are given n_j and m_j-vectors respectively and λ is a parameter in a specified range $\lambda_l \leqslant \lambda \leqslant \lambda_u$. This problem will be referred to as the Parametric Linear Primal problem (**PLP**).

Considering the partitioning introduced in Section 3, we may write a relation analogous to (3.1) as:

$$x_{j1} = B_{j1}^{-1}(b_j + \lambda e_j) - B_{j2}^{-1} x_{j2} - B_{j1}^{-1} D_j y, \tag{5.1}$$

whence we may state the Parametric Linear Modified primal problem (**PLM**) as:
Maximize

$$\alpha(\lambda) + \sum_{j=1}^{k} (d_j^1 + \lambda d_j^2)' x_{j2} + (d_0^1 + \lambda d_0^2)' y,$$

subject to the constraints

$$\sum_{j=1}^{k} M_j x_{j2} + M_0 y = b + \lambda e,$$

$$x_{j2} \geqslant 0, \qquad y \geqslant 0,$$

where

$$\alpha(\lambda) = \sum_{j=1}^{k} (c_{j1} + \lambda f_{j1})' B_{j1}^{-1}(b_j + \lambda e_j),$$

$$b + \lambda e = (b_0 + \lambda e_0) - \sum_{j=1}^{k} A_{j1} B_{j1}^{-1}(b_j + \lambda e_j),$$

$$d_0^1 + \lambda d_0^2 = (c_0 + \lambda f_0) - \sum_{j=1}^{k} (B_{j1}^{-1} D_j)' (c_{j1} + \lambda f_{j1}),$$

$$d_0^1 + \lambda d_j^2 = (c_{j2} + \lambda f_{j2}) - (B_{j1}^{-1} B_{j2})' (c_{j1} + \lambda f_{j1}).$$

Clearly, all properties of **LM** are shared by **PLM** and the two problems are equivalent for $\lambda = 0$.

We assume that an optimal solution for $\lambda = \lambda_0$ has already been obtained by the decomposition method outlined in the previous sections. The questions of post-optimality analysis to be examined here, are:

(I) For which values of the parameter λ, $\lambda_l \leqslant \lambda \leqslant \lambda_u$, $\lambda \neq \lambda_0$ does the current solution remain feasible and optimal? This question is commonly referred to as "ranging information" on the current optimal solution.

(II) For a given change in the value of the parameter λ, for which the current basic solution for $\lambda = \lambda_0$ is no longer feasible or optimal, what is the new feasible and optimal basic solution? This is usually referred to as "parametric programming."

The procedures described in this section provide solutions to these questions by using and expanding on, information available from the current optimal solution for $\lambda = \lambda_0$. The ranging information is obtained by the well known ratios (see e.g. [*8*]). The parametric programming algorithm provides mechanisms for altering the existing optimal solution, so that it remains feasible and optimal when the value of the parameter falls outside one of the computed "ranges." Thus, (1) if feasibility in the last **PLM** is violated, a basis change is performed using the dual simplex method, (2) if the nonnegativity of at least one variable $(x_{j1})_\nu$ is violated for at least one j, either the current partitioning is altered by exchanging $(x_{j1})_\nu$ by a nonbasic variable $(x_{j2})_\mu$, or the violated nonnegativity constraint expressed in terms of x_{j2} and y is appended to the last **PLM**. (3) If the optimality condition is violated, a basis change is made in the last **PLM** using the primal simplex method, provided that the resulting levels of the x_{j1} $(j = 1,..., k)$ variables are nonnegative. If not, their nonnegativity is secured by following (2) above.

The Algorithm

Let $(y(\lambda), x_{j1}(\lambda), x_{j2}(\lambda))$ $(j = 1,..., k)$ be the optimal solution to **PLP** for $\lambda = \lambda_0$ and $(y(\lambda), x_{j2}(\lambda))$ $(j = 1,..., k)$ be the solution to the last **PLM** whose definition and data are also available. The latter is assumed to have q rows, $m_0 + p + 1 \leqslant q \leqslant m_0 + p + k$, and s variables.

We now examine the postoptimality question (I), i.e.

(I) For which values of λ; $\lambda_l \leqslant \lambda \leqslant \lambda_u$, $\lambda \neq \lambda_0$ does $(y(\lambda), x_{j1}(\lambda), x_{j2}(\lambda))$ satisfy the conditions for: (1) feasibility, (2) optimality, (3) both feasibility and optimality.

For (1), we wish to determine the largest interval for which overall feasibility is maintained. We distinguish two cases.

(a) Feasibility condition on the last **PLM**: This is a necessary condition for feasibility in **PLP**. The largest interval for which this condition is satisfied, denoted by $[\lambda_{l1}^f, \lambda_{u1}^f]$, is obtained by considering the right-hand-side vector of the last **PLM** (i.e. for $\lambda = \lambda_0$) updated by the inverse of its optimal basis M_B^{-1}, i.e. we consider $(g^1 + \lambda g^2) = M_B^{-1}(b + \lambda e)$ and apply the condition

$$g^1 + \lambda g^2 \geqslant 0 \tag{5.2}$$

which gives (i) For $\lambda > \lambda_0$:

$$\lambda_{u1}^f = \min\{-(g^1)_\nu/(g^2)_\nu \mid (g^2)_\nu < 0; \quad \nu = 1,..., q\}, \tag{5.3}$$

or $\lambda_{u1}^f = +\infty$ if $(g^2)_\nu \geqslant 0$ for $\nu = 1,..., q$. (ii) For $\lambda < \lambda_0$:

$$\lambda_{l1}^f = \max\{(g^1)_\nu/(g^2)_\nu \mid (g^2)_\nu > 0; \quad \nu = 1,..., q\},$$

or $\lambda_{l1}^f = -\infty$ if $(g^2)_\nu \leqslant 0$ for $\nu = 1,..., q$.

(b) Feasibility condition on the x_{jl} variables: In addition to (a) above, it is necessary to satisfy the nonnegativity restrictions on the $x_{j1}(\lambda)$ whose levels, denoted by $g^1 + \lambda g^2$, are established by substituting the optimal levels of the basic $x_{j2}(\lambda)$ and $y(\lambda)$ into (5.1) which gives the optimal $x_{j1}(\lambda)$ as linear functions of λ, denoted by $(g_j^1 + \lambda g_j^2)$ $(j = 1,..., k)$. The largest interval $[\lambda_{l2}^f, \lambda_{u2}^f]$ for which nonnegativity of the $x_{j1}(\lambda)$ is maintained is obtained by: (i) For $\lambda > \lambda_0$:

$$\lambda_{u2}^f = \min\{-(g_j^1)_\nu/(g_j^2)_\nu \mid (g_j^2)_\nu < 0; \quad \nu = 1,..., m_j; \quad j = 1,..., k\},$$

or $\lambda_{u2}^f = \infty$ if $(g_j^2)_\nu \geqslant 0$ for all ν and j. (ii) For $\lambda < \lambda_0$:

$$\lambda_{l2}^f = \max\{(g_j^1)_\nu/(g_j^2)_\nu \mid (g_j^2)_\nu > 0; \quad \nu = 1,..., m_j; \quad j = 1,..., k\},$$

or $\lambda_{l2}^f = -\infty$ if $(g_j^2)_\nu \leqslant 0$ for all ν and j. For overall feasibility we must therefore have $\lambda \in [\lambda_{l1}^f, \lambda_{u1}^f] \cap [\lambda_{l2}^f, \lambda_{u2}^f]$.

Now for (2), we wish to determine the largest interval $[\lambda_{l1}^e, \lambda_{u1}^e]$ for which the current solution remains optimal. This is easily accomplished by considering the cost row of the last **PLM** (updated by the inverse of its optimal basis), denoted by $(h_j^1 + \lambda h_j^2)$ $(j = 0, 1,..., k)$. Thus, for optimality we must have, for the $x_{j2}(\lambda)$ variables:

$$h_{j1} + \lambda h_j^2 \leqslant 0 \qquad (j = 1,..., k), \tag{5.4}$$

and for the $y(\lambda)$ variables:

$$h_0^1 + \lambda h_0^2 \leqslant 0. \tag{5.5}$$

The sought interval is immediately established by: (i) For $\lambda > \lambda_0$:

$$\lambda_{u1}^e = \min\{-(h_j^1)_{\nu_j}/(h_j^2)_{\nu_j} \mid (h_j^2)_{\nu_j} > 0; \text{ all}^5\ \nu_j\ ; j = 0, 1,..., k\}, \tag{5.6}$$

or $\lambda_{u1}^e = +\infty$ if $(h_j^2)_{\nu_j} \leqslant 0$ for all[4] ν_j ; $j = 0, 1,..., k$. (ii) For $\lambda < \lambda_0$:

$$\lambda_{l1}^e = \max\{(h_j^1)_{\nu_j}/(h_j^2)_{\nu_j} \mid (h_j^2)_{\nu_j} < 0; \text{ all}^5\ \nu_j\ ; j = 0, 1,..., k\},$$

or $\lambda_{l1}^e = -\infty$ if $(h_j^2)_{\nu_j} \geqslant 0$ for all[5] ν_j ; $j = 0, 1,..., k$.

Finally, (3) is obtained as an obvious consequence of (1) and (2). That is, the solution will remain both feasible and optimal for $\lambda \in [\lambda_*, \lambda^*]$ where

$$\lambda_* = \max\{\lambda_{l1}^e, \lambda_{l1}^f, \lambda_{l2}^f, \lambda_l\}; \qquad \lambda^* = \min\{\lambda_{u1}^e, \lambda_{u1}^f, \lambda_{u2}^f, \lambda_u\}.$$

If $\lambda_* = \lambda_l$ and $\lambda^* = \lambda_u$ then the postoptimality question I has been answered and question II is clearly not relevant. We must assume, therefore, that either $\lambda_l < \lambda_*$ or $\lambda^* < \lambda_u$, or both. In the ensuing discussion we consider only the case $\lambda^* < \lambda_u$. This is no loss of generality since the case $\lambda < \lambda_*$ leads to entirely symmetric results.

The postoptimality question II, i.e. "parametric programming", is stated as:

(II) Utilizing the information available from the current optimal solution for $\lambda = \lambda^*$, obtain the optimal solution to **PLP** for $\lambda = \lambda^* + \epsilon$; $\epsilon > 0$.

We consider three cases: (1) $\lambda^* = \lambda_{u1}^f$, i.e. for $\lambda = \lambda^* + \epsilon$; $\epsilon > 0$ the current optimal solution to the last **PLM** does not satisfy the feasibility condition of **PLM**. (2) $\lambda^* = \lambda_{u2}^f$; i.e. for $\lambda = \lambda^* + \epsilon$; $\epsilon > 0$ the current optimal solution does not satisfy the nonnegativity condition on the $x_{j1}(\lambda)$. (3) $\lambda^* = \lambda_{u1}^e$, i.e. for $\lambda = \lambda^* + \epsilon$; $\epsilon > 0$ the current optimal solution does not satisfy the optimality conditions.

For (1), we would like to effect a basis change in the last **PLM** optimal basic solution such that the feasibility condition (5.2) will be restored with respect to the new basis.

[5] Assuming (for notational purposes) that the B_{j1} are formed by the first m_j columns of B_j for $j = 1,...,k$, "all ν_j" refers to:

$$\nu^0 = 1,..., p;\ \nu_j = m_j + 1,..., n_j\ ; j = 1,..., k.$$

This is easily accomplished by considering the optimal levels of the basic $y(\lambda)$ and $x_{j1}(\lambda)$ variables (earlier denoted as $g^1 + \lambda g^2$) for $\lambda = \lambda_{u1}^f$. Due to (5.3) we must have $(g^1 + \lambda g^2)_\nu = 0$ for at least one component ν. For $\lambda = \lambda_{u1}^f + \epsilon$, therefore, we have a basic optimal but infeasible solution. The customary rules of the dual simplex method (see, e.g., p. 247 in [*8*]) are applied and the νth variable is exchanged with one of the nonbasic x_{j1} and y variables which enters the basis at zero level. This requires exactly one pivot step when ν is unique. Several pivot steps may be necessary to obtain feasibility if ν is not unique. If no nonbasic variable is eligible to enter, i.e. row ν of the last **PLM** simplex tableau has no negative entries, we conclude that there is no feasible solution to **PLM**, further implying that no such solution to **PLP** exists.

For (2) we assume that for $\lambda = \lambda_{u2}^f$ we obtain from (5.1) $x_{j1} \geqslant 0$ with $(x_{j1})_\mu = 0$ for at least one j and μ. For $\lambda = \lambda_{u2}^f + \epsilon$; $\epsilon > 0$ we wish to restore the nonnegativity of the $x_{j1}(\lambda)$. This may be accomplished by an exchange between the $x_{j1}(\lambda)$ and $x_{j2}(\lambda)$ variables; that is by updating the current partitioning of the problem, or by appending an additional constraint to **PLM**. For a fixed value of $\lambda = \lambda_{u2}^f$ and the corresponding optimal solution to the last **PLM,** (5.1) gives the form: $x_{j1} = p_j - P_j x_{j2}$ $(j = 1, \ldots, k)$ where $P_j = B_{j2}^{-1} B_{j2}$; $p_j = B_{j1}^{-1}(b_j + \lambda e_j) - B_{j1}^{-1} D_j y(\lambda)$. At this point we treat two cases:

(i) If there exists a column index ν such that $(P_j)_{\mu\nu}(x_{j2})_\nu \neq 0$ where $(P_j)_{\mu\nu}$ is the element in position (μ, ν) of P_j, then the variables $(x_{j1})_\mu$ and $(x_{j2})_\nu$ are exchanged. The current partitioning is updated to reflect this exchange, a revised **PLM** is defined and solved to optimality. The computational effort required to solve this revised **PLM** may be drastically reduced by attempting to use as a starting basic set those column indices which were in the optimal basic set of the previous **PLM.**

(ii) If $(P_j)_{\mu\nu}(x_{j2})_\nu = 0$ for all ν, the above exchange is not possible. Nevertheless, we can secure the nonnegativity of $(x_{j1})_\mu$ by generating and appending an "additional constraint" of the form (3.6) expressing this restriction in terms of the x_{j2} and y. An optimal solution to this enlarged **PLM** is then obtained by revising the optimal solution to the current **PLM** by the well known rules (see, e.g., pp. 384–385 in [*8*]).

The above two cases lead to the consideration of a solution strategy whereby one may keep applying (ii) until either the number of additional constraints becomes excessive, or case (i) is possible. That is (i) may be used at will, whenever possible, to reduce the size of the **PLM** by eliminating all of the accumulated "additional constraints." The number of "additional constraints" may also be reduced while applying exclusively case (ii), by omitting such constraints as soon as they become inactive. If they become active at later stages the appropriate "additional constraint" will be generated (case (ii)), or alternately the nonnegativity of the corresponding x_{j1} variable will be guaranteed by a revised partitioning (case (i)).

Finally, for (3), we would like to effect a basis change such that the optimality conditions (5.4)–(5.5) are restored. For $\lambda = \lambda_{u1}^{e}$, (5.6) guarantees the existence of at least one nonbasic variable $(z)_\nu$, among the components of $x_{j2}(\lambda)$ or $y(\lambda)$, with a reduced cost of zero. The rules of the simplex method could be applied to introduce $(z)_\nu$ into the basis. However, due to the provisional nature of this pivot step (see case b below), we first examine the effect of such a step on the current levels x_{j1}^{b} of the variables x_{j1}.

For restoring feasibility in **PLM**, we must have, at the conclusion of the pivot step: $z_B{}^a = z_B{}^b - p_\nu{}^b(z^b)_\nu \geqslant 0$, where $z_B{}^b$ and $z_B{}^a$ are the basic optimal solution vectors before and after the pivot step respectively and $p_\nu{}^b$ is the column corresponding to the nonbasic variable $(z^b)_\nu$ in the current simplex tableau of **PLM**. The effect of the pivot step would thus be to increase the level of $(z)_\nu$ from zero to:

$$(z^a)_\nu = \min_i\{(z_B)_i/(p_\nu{}^b) \mid (p_\nu{}^b)_i > 0; \qquad i = 1,\dots, g\}$$

and $(z_B{}^a)_\rho = 0$ for at least one component ρ. If, upon substituting $z_B{}^a$ into (5.1), the resulting x_{j1}^{a} are strictly positive, then the contemplated pivot step, with (μ, ν) as the pivot position, is carried out.

Alternately, if $(x_{j1}^{a})_\rho = 0$ for at least one ρ and:

(a) $x_{j1}^{a} \geqslant x_{j1}^{b}$, then the contemplated pivot step is performed.

(b) $x_{j1}^{a} < x_{j1}^{b}$, then the nonnegativity of $x_{j1}(\lambda)$ for $\lambda = \lambda_{u1}^{e} + \epsilon$, $\epsilon > 0$, is secured by the procedure outlined in 2(i)–(ii) above.

It should be noted however, that if **PLM** is solved by the product form of the inverse revised simplex method, it is computationally expedient to carry out the pivot step in advance and subsequently check its validity. If (b) prevails, return to the pre-pivot status of **PLM** is achieved by simply dropping the last elementary matrix in the product form of the inverse.

6. Verification of the Method

In this section the validity of the algorithm is outlined.

Suppose that in the tth cycle the problem **M** has s_t "additional constraints" of the form (3.6). It follows from (3.1) that **M** is equivalent to the problem (2.1)–(2.3) and the constraints

$$y \geqslant 0, \qquad x_{j2} \geqslant 0 \qquad (j = 1,\dots, k),$$

and

$$(x_{j_\nu 1})_{i_\nu} \geqslant 0 \qquad (\nu = 1,\dots, s_t), \tag{6.1}$$

where $(x_{j_\nu}1)_{i_\nu} \geqslant 0$ corresponds to the νth constraint of the form (3.6).

Since canceling the restriction $(x_j)_\nu \geqslant 0$ $(j = 1,\dots, k)$ in the primal problem (**P**) results in the removal of the column corresponding to $(w_j)_\nu$ from the dual problem

(2.5)–(2.8), it follows that in the second formulation of the dual (i.e. in **D** or for the linear case in **LD**) the νth inequality in the jth block is replaced by an equation. Therefore, the dual problem of the problem stated by (2.1)–(2.3) *and* (6.1) is given by:

Minimize

$$\Phi(y, x, u) = F(y, x_1, \ldots, x_k) + \sum_{j=0}^{k} b_j' u_j - \nabla F(y, x_1, \ldots, x_k)' \cdot (y, x_1, \ldots, x_k), \quad (6.2)$$

subject to:

$$\sum_{j=1}^{k} D_j' u_j + D_0' u_0 - \nabla_y F(y, x_1, \ldots, x_k) \geqslant 0, \quad (6.3)$$

$$\left.\begin{aligned} &B_{j1}' u_j + A_{j1}' u_0 - \nabla_{x_{j1}} F(y, x_1, \ldots, x_k) \overset{(\geqslant)}{=} 0, \\ &B_{j2}' u_j + A_{j1}' u_0 - \nabla_{x_{j2}} F(y, x_1, \ldots, x_k) \geqslant 0, \end{aligned}\right\} \quad (j = 1, \ldots, k) \quad (6.4)$$

Similarly, for the linear case, the above dual problem is given by:

Minimize

$$\sum_{j=1}^{k} b_j' u_j, \quad (6.2a)$$

subject to

$$\sum_{j=1}^{k} D_j' u_j + D_0' u_0 \geqslant c_0, \quad (6.3a)$$

$$\left.\begin{aligned} &B_{j1}' u_j + A_{j1}' u_0 \overset{(\geqslant)}{=} c_{j1}, \\ &B_{j2}' u_j + A_{j2}' u_0 \geqslant c_{j2}, \end{aligned}\right\} \quad (j = 1, \ldots, k) \quad (6.4a)$$

where $\overset{(\geqslant)}{=}$ means that the constraints corresponding to the variables $(x_{j,1})_{i_\nu}$ $(\nu = 1, \ldots, s_t)$ are inequalities.

The following theorem states the optimality condition:

THEOREM 1. *Let* $(y^*, x_{j1}^*, x_{j2}^*)$ $(j = 1, \ldots, k)$ *be the vector obtained after t cycles. It is an optimal solution to* **P** *if and only if* $x_{j1}^* \geqslant 0$ $(j = 1, \ldots, k)$.

Proof. The condition is clearly necessary because otherwise (2.4) would not be satisfied. For sufficiency, we note from (3.1) that $(x_{j1}^*, x_{j2}^*, y^*)$ is an optimal solution to the problem given by (2.1)–(2.3) and (6.1). If $x_{j1}^* \geqslant 0$ $(j = 1, \ldots, k)$ the condition (6.1) can be replaced by (2.4) without changing the optimal solution. Thus, $(x_{j1}^*, x_{j2}^*, y^*)$ is an optimal solution to **P** if $x_{j1}^* \geqslant 0$. ∎

In order to prove that an optimal solution to **P** is obtained after a finite number of cycles, we need the following two statements.

LEMMA 1. *To each vector* $(y^*, x_{j1}^*, x_{j2}^*)$ $(j = 1,..., k)$, *obtained in the* tth *cycle, there exists a corresponding vector* $(y^*, x_j^*, u_0^*, u_j^*)$ $(j = 1,..., k)$ *which is a feasible point of* **D** *and has the property*:

$$F(y^*, x_1^*,..., x_k^*) = \Phi(y^*, x_1^*,..., x_k^*, u_0^*, u_1^*,..., u_k^*), \tag{6.5}$$

where $x_j^* = (x_{j1}^*, x_{j2}^*)$.

Proof. $(y^*, x_{j1}^*, x_{j2}^*)$ $(j = 1,..., k)$ is an optimal solution to the problem given by (2.1)–(2.3) and (6.1). Since (6.2)–(6.4) define the dual of this problem, it follows from the duality theorem for nonlinear programming [*21*] that there exists a point $(y^*, x_j^*, u_0^*, u_j^*)$ $(j = 1,..., k)$ satisfying (6.3)–(6.4) such that the objective functions (2.1) and (6.2) have equal values, immediately establishing the property (6.5). Comparison of (6.3)–(6.4) with (2.10)–(2.11) shows that each feasible point of (6.3)–(6.4) is also a feasible point of (2.10)–(2.11) (but not conversely). ∎

For a linear objective function (6.5) is simply:

$$\sum_{j=1}^{k} c_j' x_j^* + c_0' y^* = \sum_{j=0}^{k} b_j' u_j^*. \tag{6.6}$$

LEMMA 2. *Let* $(y^*, x_{j1}^*, x_{j2}^*)$ *and* $(y^{**}, x_{j1}^{**}, x_{j2}^{**})$ $(j = 1,..., k)$ *be the vectors obtained at the* tth *and* $(t + 1)$th *cycle, respectively. Then,*

$$F(y^{**}, x_1^{**},..., x_k^{**}) \leqslant F(y^*, x_1^*,..., x_k^*) \tag{6.7}$$

Proof. In the tth cycle we have solved a problem given by (2.1)–(2.3) and (6.1). Denote the feasible region of this problem by R_1. This domain is subsequently altered, according to the procedure described in Section 3, as follows:

(1) If **M** contains "additional constraints" of the form (3.6) we cancel those which are not active in the optimal solution (Case 1). Each remaining "additional constraint" is either left unchanged (Case 3) or rewritten (Case 2) while one of the constraints $(x_{j2})_i \geqslant 0$ (which is not active in the optimal solution since $(x_{j2}^*)_i > 0$) is disregarded.

(2) Suppose x_{j1}^* has at least one negative component, say $(x_{j1}^*)_\nu$. If Case I applies, one of the constraints $(x_{j2}^*)_i \geqslant 0$ (which is inactive since $(x_{j2}^*)_i > 0$) is canceled and replaced by $(x_{j1}^*)_\nu \geqslant 0$. If Case II applies, an additional constraint of the form (3.6), equivalent to $(x_{j1})_\nu \geqslant 0$, is added to the problem.

Thus, upon completion of a cycle, say the tth, only inactive constraints are canceled while the new "additional constraints" of the form (3.6) which are added to **M** are not satisfied by $(y^*, x_{j1}^*, x_{j2}^*)$. For a maximization problem this implies (6.7). ∎

For a linear objective function (6.7) is simply:

$$\sum_{j=1}^{k} c_j' x_j^{**} + c_0' y^{**} \leqslant \sum_{j=1}^{k} c_j' x_j^{*} + c_0' y^{*}. \tag{6.8}$$

Remark 1. The above proof shows that the feasible domain R_1 is altered in two steps. First, by canceling some constraints, we obtain a larger domain R_2 in which the objective function remains at its optimal solution value. Then, new constraints (i.e. nonnegativity restrictions on the x_{j2} variables) which are not satisfied by the current optimal solution are added. This results in a smaller feasible domain, say R_3. It follows, therefore, that strict inequality holds in (6.7) and (6.8), except in the case of an alternate optimal solution in R_3. In this case, a possibility of cycling exists. Nevertheless, it can easily be prevented by a small perturbation in the coefficients of (2.1) or in the c_j; $(j = 0, 1,..., k)$ for the linear case. Clearly, cycling will not occur for strictly concave objective functions since in such cases the optimal solution is unique.

Remark 2. In Section 3 we assumed that if a feasible solution to **P** exists, then **M** attains an optimal solution for a finite point $(y, x_{12},..., x_{k2})$. Now, suppose that the latter is not true, i.e. **M** does not attain an finite optimum. In order to prevent such occurrences, we propose the following procedure.

Let T be a sufficiently large positive number, and o_j, q_j vectors, conformal to the current partitioning of $x_j = (x_{j1}, x_{j2})$, which have as their components all zeros and ones respectively. Then, the addition of the condition $\sum_{j=1}^{k} q_j' x_{j2} + q_0' y \leqslant T$ to the existing constraints of **M**, insures that this enlarged **M** has an optimal solution provided that **M** has a feasible solution. Clearly, this is equivalent to the addition of:

$$\sum_{j=1}^{k} (o_j, q_j)' \binom{x_{j1}}{x_{j2}} + q_0' y + \tau = T; \qquad \tau \geqslant 0$$

to the constraints of **P**. If in the optimal solution to this enlarged **P** we have $\tau = 0$ for arbitrary large T, then the original problem has an unbounded solution.

Since the optimal value of the current **M** is an upper bound to the objective function values of all subsequent **M** problems (Lemma 2), and due to the way in which the feasible domain of **M** is altered from one cycle to the next, it follows that all subsequent **M** problems have optimal solutions, provided they have a feasible solution.

THEOREM 2. *If* **P** *has an optimal solution it is obtained in a finite number of cycles.*

Proof. Let $(y^*, x_j^*, u_0^*, u_j^*)$ and $(y^{**}, x_j^{**}, u_0^{**}, u_j^{**})$ $(j = 1,..., k)$ be the feasible points of **D** associated with the vectors $(y^*, x_{j1}^*, x_{j2}^*)$ and $(y^{**}, x_{j1}^{**}, x_{j2}^{**}$ $(j = 1,..., k)$ obtained in the tth and $(t + 1)$th cycle, respectively (Lemma 1). Considering (6.5) in conjunction with Lemma 2 we see that:

$$\Phi(y^{**}, x_{j1}^{**}, x_{j2}^{**}, u_0^{**}, u_j^{**}) \leqslant \Phi(y^*, x_{j1}^*, x_{j2}^*, u_0^*, u_j^*) \qquad (j = 1,..., k). \tag{6.11}$$

We observe that $(y^*, x_j^*, u_0^*, u_j^*)$ $(j = 1,..., k)$ is an optimal solution to the dual problem (6.2)–(6.4) and that it satisfies as equalities at least those constraints in (6.3)–(6.4) which correspond to cancelation of the nonnegativity restrictions on the x_{j1} in **M**. Denote the set of equations in (6.3)–(6.4) for the tth and $(t + 1)$th cycle by S^* and S^{**}, respectively. If (6.11) is an equality for several consequtive cycles, appropriate methods to prevent cycling can be employed to insure that S^* re-occurs at most a finite number of times.

If **D** has an unbounded solution, it follows that a problem (6.2)–(6.4) with an unbounded solution is obtained after a finite number of cycles. Since $G(y, x_{j2})$ is differentiable, by the duality theory for nonlinear programming [*21*], this implies that **M** has no feasible solution. The latter then implies that **P** has no feasible solution.

If **D** has an optimal solution, then it follows from the preceding discussion that it is obtained in a finite number of cycles. If **D** satisfies a constraint qualification (which is satisfied if, e.g., $F(y, x_j)$ is strictly concave), the converse duality theorem [*10*] asserts that the optimal solution of the corresponding **M** yields the optimal solution to **P**. Alternately, for cases where the constraint qualification is not satisfied, the optimal solution to the corresponding **M** need not be feasible for **P**. However, since we have an optimal solution to **D**, it follows from Lemmas 1 and 2 that any subsequent **M** problem has either no feasible solution or an optimal solution for which the objective function has a value equal to the optimal value of **D**. Hence, using appropriate methods to prevent cycling (Remark 1) we arrive, after a finite number of additional cycles, at an **M** which either provides an optimal solution to **P** or has no feasible solution. ∎

Remark 3. For the case of a linear objective function the above proof may be stated in a concise manner as follows:

The relation (6.11) implies:

$$\sum_{j=1}^{k} (c'_{j1}x_{j1}^{**} + c'_{j2}x_{j2}^{**}) + c_0'y^{**} \leqslant \sum_{j=1}^{k} (c'_{j1}x_{j1}^{*} + c'_{j2}x_{j2}^{*}) + c_0'y^{*}. \tag{6.12}$$

Therefore, with appropriate methods to prevent cycling, the optimal solution of **LD**, if it exists, will be reached in a finite number of cycles, say after r cycles. By the duality theorem for linear programming, the vector $(x_{j1}^r, x_{j2}^r, y^r)$ $(j = 1,..., k)$ obtained in the rth cycle is an optimal solution to **LP**. If **LD** has no optimal solution, it follows again from the duality theorem that **LP** also has no optimal solution.

7. Computational Aspects and Results

The computational efficiency of the algorithm presented in Section 3 depends on several factors. First, the distinction between linear and nonlinear objective function is an essential one. It is generally known that for most of the available methods, the solution efficiency for nonlinear problems depends almost entirely on the number of variables. This is particularly evident when one uses a method in the dual space such as Gradient Projection [15]. Consequently, the reduction in the total number of variables involved in each **M** solution should be viewed as a much more important development than the obvious reduction in the number of constraints. This reduction can be impressive for many problems arising in practical applications. Then, little attention is paid to the increase in "additional constraints" of the form (3.6) during the course of the algorithm. Their accumulation is tolerated and their elimination may be deferred until convenient.

The situation may be markedly different for the linear case depending on the method of solving **LM**. Its solution may be accomplished either by the primal or the dual simplex method. The choice will depend on the size of **LM** which is related to the size of the original problem **LP**. If this problem is specified with subproblem matrices B_j for which $m_j \ll n_j$, then the number of variables in **LM** will still be substantial, thus dictating the use of the primal simplex method for its solution. The accumulation of additional constraints will then be checked by effective pivoting procedures. As mentioned earlier, however, the existence of nonvanishing pivots cannot be guaranteed for all the variables corresponding to existing "additional constraints." Therefore, even with the emphasis on pivoting, the possibility of a modest accumulation of these constraints remains. On the other hand, if $m_j \leqslant n_j$ with $m_n \approx n_j$, the number of x_{j2} variables in **LM** will be relatively small. In such instances, use of the dual simplex method should prove more efficient. In this case, solution of **LM** to optimality may be avoided (see, e.g., Theorem 4 in [16]. The number of additional constraints may then be allowed to increase more freely, with pivoting assuming a secondary role.

The choice of initial bases B_{j1} for the subproblem matrices B_j is an obvious parameter which affects solution efficiency. Clearly, the optimal solution to the complete problem **P** would be obtained in one cycle if this choice were made to coincide with the optimal basis. In most industrial problems an initial point (y^0, x_j^0), $j = 1,..., k$ (not necessarily feasible) will be known from the physical characteristics of the model or from a previous solution to a slightly modified problem. The columns of B_j which correspond to the positive components of the x_j^0 specify the partial initial basis which may then be used, whenever linear independence holds, to construct the inverse B_{j1}^{-1} by appending, if necessary, some linearly independent nonbasic columns. Other methods of obtaining an initial subproblem basis may be more advantageous. However, computational evidence will be required to establish their relative merits.

The solution efficiency will also be influenced by the method of variable exchanges, referred to as "pivoting." Such exchanges are required under both steps A and B of the algorithm. Complete lack of nonzero pivots at the required positions will cause the generation of at least one "additional constraint" for the next cycle. Since generation of an excessive number of such constraints is undesirable, at least when **LM** is solved by the primal simplex method, intuition suggests that more than one pivot step should be performed for the nonoptimal blocks at each cycle. One way of performing this operation is to apply the simplex method to a modified subproblem as follows. Let the current basis for the jth subproblem be B_{j1}, and let the submatrix of B_j containing the nonbasic columns be B_{j2}. Suppose that the solution of $\mathbf{M}$ and the subsequent application of (3.1) gives the following partition of variables $x_{j1} = (x_{I_1}, x_{I_2}, x_{I_3})$; $x_{j2} = (x_{I_4}, x_{I_5})$ with

$$\begin{aligned} I_1 &= \{i \mid (x_{j1})_i < 0\}; \\ I_2 &= \{i \mid (x_{j1})_i = 0;\ i \in I_a\}; \\ I_3 &= \{i \mid (x_{j1})_i \geqslant 0\}; \\ I_4 &= \{i \mid (x_{j2})_i > 0\} \end{aligned}$$

where I_a represents the set of column indices i for which "additional constraints" of the form (3.6) were present in $\mathbf{M}$. An effective pivoting strategy would then be to exchange as many of the variables in I_1 and I_4 as possible, and to retain the variables I_3 as basic. This is accomplished by considering the linear program:

Maximize

$$-T_1 q'_{I_1} x_{I_1} - T_2 q'_{I_2} x_{I_2} + T_4 q'_{I_4} x_{I_4}, \tag{7.1.1}$$

subject to

$$B_{I_1} x_{I_2} + B_{I_2} x_{I_2} + B_{I_3} x_{I_3} + B_{I_4} x_{I_4} = \mathbf{0}, \tag{7.1.2}$$

$$x_{I_1}, x_{I_2}, x_{I_3}, x_{I_4} \geqslant 0, \tag{7.1.3}$$

where $q_{I_1}, q_{I_2}, q_{I_4}$ are vectors having all ones in their components and the scalars $T_1, T_2, T_4 > 0$ are specified weighing constants. This problem is solved by the primal simplex method. In order to retain x_{I_3} in the basis, the usual pivot selection rules of the simplex method are revised to avoid pivoting x_{I_3} out of the basis. In our program, an initial inverse for the above problem is obtained by reinverting the subproblem basis of the previous decomposition cycle. It would certainly be more efficient to maintain each inverse B_{j1}^{-1} in the product form which can then be revised, if the block is nonoptimal, by the simplex algorithm. If $T_1 \ll T_2 = T_4$, then the solution to (7.1.1)–(7.1.3) will obtain the revised subproblem basis B_{j1} and its inverse by following the best pivoting sequence with preference given to reducing the infeasibility caused by x_{I_1}. Thus, the exchanging will take place primarily between x_{I_1} and x_{I_4} and only to a limited degree between x_{I_2} and x_{I_4}. If the weighing constants are chosen so that $T_2 \ll T_1 = T_4$, then exchanging will favor the elimination of the existing additional constraints over the reduction in the existing infeasibility. It is reasonable to assume then that the choice of these constants will also influence the overall efficiency. However, limited computational experience in comparing the two extreme choices stated above, indicated no appreciable differences in the number of cycles.

Finally, the choice of an initial starting point for each **M** is the key to the overall efficiency. Starting each **M** from the solution to the previous **M** seems to be a plausible way. Although such a point will be infeasible for the **M** of the current decomposition cycle, the method outlined in [*15*] may be applied to obtain a feasible starting point. It is expected that this choice will be a good one, particularly in the later decomposition cycles. Similarly, for the linear case, the optimal basis to the previous **LM** will provide a partial basis for starting the solution to the current **LM**. Our experience has shown that the optimal bases of successive **LM** problems differ from each other only by a few basic columns. This observation leads us to expect that the use of the previous basis columns, which are still present in the new **LM**, as a partial starting basis,will result in considerable computational savings.

A small experimental computer program for solving the linear problem **LP** has been written in FORTRAN and has been tested on a number of randomly generated problems.

The program, which is completely core resident, is divided into a number of subroutines which essentially perform the following functions: (a) Problem data input or generation of input data, (b) Construction of initial bases B_{j1} ; $j = 1,..., k$ and their inverses, (c) Generation of the **LM** matrices, (d) Solution of **LM**, (e) Extraction of solution values, computation of x_{j1} and optimality tests, (f) Variable exchanges for the subproblems, and (g) Output and solution check.

The input phase reads in the matrices A_j, B_j, D_0, D_j, the vectors b_0, b_j, c_0, and c_j ($j = 1,..., k$). Optionally, these matrices and vectors are generated randomly with the necessary precautions insuring feasibility and boundedness. This part of the program also generates the input data for the complete problem **LP** in a form acceptable by the CDM4-LP System [*22*] for the CDC3600. Subsequently this data is used to obtain solutions for each **LP** as an ordinary linear program. The initial bases, for this early version of the program, are taken as unit matrices representing slack and artificial vectors. The bulk of the computational work is done in (c) above, where the matrices, cost and right hand side vectors of **LM** are computed and are stored in packed form for later use by CDM4 which is then called to solve **LM** starting from a completely artificial basis. This undesirable manner of starting the solution of **LM** was chosen in the interest of simplicity since it was found that CDM4 could not effectively handle a given partial starting basis. The solution values, basis, etc. obtained by the CDM4 are extracted by unpacking. Then, the current value of the objective function and the levels of x_{j1} ; $j = 1,..., k$ are computed. For each nonoptimal block, the necessary variable exchanges are performed by direct pivoting, or optionally, by solving the modified subproblem (7.1.1)–(7.1.3) by CDM4. The data for these problems are packed and set-up for use by the CDM4 using the subroutines of (c) above. Each problem (7.1.1)–(7.1.3) is solved by first reinverting to the previous (feasible) B_{j1} and then by carrying Phase II iterations. Each block is handled in succession and requires the original data of B_j only. The output phase is negligible since the (x_{j1}^*, x_{j2}^*, y) $(j = 1,..., k)$ are available from the last **LM** solution and the computation (3.1) which was necessary for the optimality test. Finally, a solution check is made by substituting the optimal variable levels into (2.2)–(2.4) to obtain a computed right-hand-side vector and by comparing it to the given b_j $(j = 0, 1,..., k)$.

A number of test cases were solved successfully by our experimental program. The data for these problems are randomly generated as follows. The matrices B_j and A_j, D_j are 50 and 100% dense, respectively with the nonzero elements of B_j arranged in a checkerboard pattern. Each element is a pseudo-random number in the range [0, 10]. In addition, unit matrices of appropriate dimensions are appended to A_j and B_j $(j = 1,..., k)$ representing nonnegative slack variables. The vectors c_j, b_j $(j = 0, 1,..., k)$ are generated by first constructing a known optimal solution to the complete problem. The desired optimal levels of the variables x_j and y are chosen so that: (a) a specified number of the coupling constraints are active (b) a specified number of the coupling variables y are at a positive level and (c) a specified number of the block variables x_j are at a positive level. Within these restrictions, randomly selected subsets of the x_j and y variables are then declared basic by assigning random levels in the range [0, 1] and random cost elements which are appropriately magnified to insure that these variables will remain basic in the optimal solution. The right-hand sides are then obtained by multiplying the constraint matrix by the generated values of x_j and y. The problem thus generated is insured to require a reasonable amount of work for its solution. The computed answers, however, may be slightly different from the generated ones for obvious reasons.

No claims will be made regarding the resemblance of these test cases to actual industrial problems. In fact, our test cases are too small to allow any inference for problems of giant sizes for which this method is primarily intended. Thus, the results of the 26 test cases presented in Fig. 7.1 should be regarded only as an indication that our method should not be abandoned. Tests of large problems of practical importance, a sophisticated computer program with the flexibility for introducing the various solution strategies discussed in the previous paragraphs and a large amount of computing time, will be needed before the efficiency of this method, or any other method previously proposed by others, may be accepted on firm ground. Plans for designing such a system and performing extensive large scale testing are reported under consideration [*20*].

The information given in Fig. 7.1 is arranged as follows:

Columns 1–6: Test case identification and problem sizes

Column 7: The number of decomposition cycles to optimality, which also corresponds to the number of **LM**'s solved

Column 8: Subproblem pivoting strategy used:

TEST CASE NO.	NO. OF COUPLING CONSTRAINTS	NO. OF COUPLING VARIABLES	BLOCK SIZE	NO. OF BLOCKS	TOTAL PROBLEM SIZE	SOLUTION BY DECOMPOSITION										DIRECT LP SOLUTION	
						NO. OF DEC. CYCLES	PIVOTING STRATEGY	AVER. NO. OF ADDL CONSTR. PER CYCLE	NUMBER OF SIMPLEX ITERATIONS FOR LM PROBLEMS		NUMBER OF VARIABLE EXCHANGES IN SUBS.		AVER. NO. OF OPTIMAL BLOCKS	TOTAL TIME FOR LM (SECS) *	TOTAL SOLUTION TIME (SECS)*	LP ITERS.	SOLUTION TIME(SECS)*.
									AVER.	TOTAL	AVER.	TOTAL					
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	5	5	10X20	2	25X45	8	1	5.0	10.7	86	2.3	18	1.5	4.6	10.3	28	11.0
						3	2	4.7	15.6	47	5.7	17	0.0	1.9	4.7		
2	10	5	10X20	2	30X45	3	1	5.7	11.3	34	1.0	3	0.5	1.4	3.0	15	6.0
						3	2	2.5	16.4	49	5.3	16	0.7	2.0	4.9		
3	5	10	10X20	2	25X50	8	1	7.7	14.4	115	1.9	15	1.0	7.2	13.8	30	8.0
						6	2	7.1	18.7	112	2.3	14	0.2	6.6	12.2		
4	10	10	10X20	2	30X50	4	1	4.2	17.5	70	1.3	5	0.5	3.8	7.1	29	9.0
						4	2	2.6	19.3	77	4.5	18	0.2	3.8	8.5		
5	5	5	10X30	2	25X65	7	1	6.3	16.9	117	3.0	21	0.1	9.0	18.3	28	10.0
						8	2	5.8	19.4	155	3.4	27	0.3	11.6	20.9		
6	10	5	10X30	2	30X65	9	1	5.4	18.4	166	2.4	22	0.2	14.1	27.7	31	12.0
						4	2	4.8	19.0	76	4.3	17	0.3	5.8	11.9		
7	5	10	10X30	2	25X70	7	1	10.4	22.3	156	2.6	18	0.1	15.1	25.2	38	13.0
						6	2	9.1	24.0	144	2.8	17	0.2	13.3	20.9		
8	10	10	10X30	2	30X70	5	1	8.2	27.2	136	2.6	13	0.2	12.6	21.2	32	13.0
						5	2	5.8	33.5	168	3.4	17	0.4	16.1	20.1		
9	5	5	15X30	2	35X65	13	1	6.4	14.9	194	3.2	41	0.0	12.7	40.7	45	13.0
						10	2	6.8	17.3	173	2.6	26	0.2	11.1	25.5		
10	10	5	15X30	2	40X65	7	1	5.7	17.4	122	2.6	18	0.3	8.6	23.7	32	12.0
						4	2	4.0	25.0	100	7.3	29	0.0	12.9	14.6		
11	5	10	15X30	2	35X70	8	1	11.1	23.2	186	2.8	22	0.1	15.7	34.5	49	15.0
						6	2	7.6	21.2	127	3.5	21	0.2	9.8	19.0		
12	10	10	15X30	2	40X70	6	1	8.6	21.2	127	3.3	20	0.2	10.9	26.3	36	15.0
						4	2	6.2	28.0	112	3.5	14	0.3	8.9	16.8		
13	5	5	10X20	5	55X105	13	1	8.4	19.6	255	5.2	68	0.9	24.0	44.3	86	45.0
						7	2	9.3	23.4	164	5.4	38	1.3	14.7	28.2		
14	10	5	10X20	5	60X105	9	1	8.9	24.2	268	5.5	49	0.3	21.9	39.7	64	24.0
						6	2	9.3	29.2	175	6.0	36	0.5	17.9	32.0		
15	5	10	10X20	5	55X110	10	1	12.0	27.4	274	5.8	58	0.5	31.6	54.8	81	32.0
						6	2	13.0	34.4	204	6.5	39	0.5	23.6	36.4		
16	10	10	10X20	5	60X110	9	1	13.0	35.4	329	5.1	46	0.7	43.4	68.2	72	31.0
						9	2	11.8	39.2	352	6.1	55	1.1	45.4	69.0		
17	5	5	10X30	5	55X155	18	1	8.6	25.5	459	5.2	93	0.6	70.4	120.0	112	52.0
						11	2	11.8	35.4	390	5.5	60	0.6	63.9	92.4		
18	10	5	10X30	5	60X155	15	1	8.6	37.0	555	6.4	96	0.5	103.5	162.0	109	55.0
						12	2	12.2	43.0	516	7.4	89	0.8	100.7	142.8		
19	5	10	10X30	5	55X160	16	1	12.3	37.4	597	5.3	85	0.6	108.2	160.9	109	55.0
						14	2	12.8	38.8	544	6.7	93	0.8	96.1	140.6		
20	10	10	10X30	5	60X160	11	1	12.0	44.6	491	6.7	74	0.5	104.4	155.5	109	71.0
						10	2	10.3	44.5	445	9.0	90	0.7	89.6	130.7		
21	5	5	15X30	5	80X155	25	1	9.5	24.1	578	6.9	172	0.2	74.0	189.2	121	63.0
						17	2	13.3	36.4	617	7.7	131	0.4	83.8	145.8		
22	10	5	15X30	5	85X155	21	1	9.0	30.8	646	6.3	132	0.8	97.4	208.9	90	52.0
						14	2	14.1	45.1	632	4.8	67	1.6	108.0	165.2		
23	5	10	15X30	5	80X160	20	1	13.4	32.8	656	5.9	117	0.5	98.8	205.4	118	66.0
						19	2	13.9	44.6	850	8.2	156	0.5	133.3	208.0		
24	10	10	15X30	5	85X160	17	1	13.4	42.6	723	6.5	111	0.4	129.1	246.1	122	72.0
						14	2	11.6	41.5	582	9.7	136	1.0	97.7	169.7		
25	5	5	10X20	10	105X205	33	1	13.0	31.6	1041	6.7	219	3.9	171.5	264.2	126	79.0
						18	2	18.0	45.8	828	5.7	103	3.3	152.0	210.1		
26	5	5	10X20	15	155X305	57	1	14.9	34.8	1984	6.0	343	7.5	457.9	659.2	199	177.0

* CDC3600 seconds

(Maximum row error is less than 2.4×10^{-7} for all solutions)

Figure 7.1 — Test results

(1): Select one component of x_{j1} corresponding to one active "additional constraint" for a particular block and exchange it with a positive component of x_{j2} for the same block. If no nonzero pivot is found, this exchange is abandoned and this "additional constraint" is left in **LM**. Select one negative component of x_{j1}, say $(x_{j1})_\nu$, for the same block and exchange it with a positive component of x_{j2} for the same block. If no nonzero pivot is found, an "additional constraint" for $(x_{j1})_\nu$ is generated for inclusion in the next **LM**.

(2): Multiple pivoting by defining and solving, for each non-optimal block, the subproblem (7.1.1)–(7.1.3) with $T_1 = 1$, $T_2 = T_4 = 100$.

Column 9: Average number of "additional constraints" which are present in **LM** at each decomposition cycle.

Columns 10, 11: The average and total number of simplex iterations required to solve **LM** by the CDM4-LP System. Each **LM** solution is started "from scratch" i.e. from a full artificial or slack basis.

Columns 12, 13: The average and total number of variable exchanges performed in the subproblems. For strategy (2) the reported numbers represent the number of simplex iterations *after* the completion of the reinversion to the current basis B_{j1}. For each nonoptimal block, reinversion generally requires an additional m_j pivot steps.

Column 14: The average number of optimal blocks in each cycle. This number is appreciably greater for problems with a large number of blocks. When a block is optimal, the corresponding matrices of **LM** remain the same for the next cycle. Our program, however, takes little advantage of this and thus some recomputation occurs.

Column 15: *Net* computation time required to solve the number of **LM** problems (given in Column 7), starting each time from a full artificial or slack basis and using the primal simplex method as programmed in the CDM4-LP System.

Column 16: The total solution time required to solve **LP** by the decomposition method starting from a full slack basis. This result includes: Preparation of **LM** matrices and vectors, packing of these for use by CDM4, solution (from scratch) of **LM**, unpacking of answers, computation of x_{j1}, optimality checks, subproblem pivoting (in case of stategy (2) preparation of the data for (7.1.1)–(7.1.3) in packed form, reinversion, solution by CDM4, unpacking of answers, etc.), generation of "additional constraints," etc.

Columns 17, 18: Number of primal simplex iterations and *net* computation time required to solve the complete problem **LP** (Column 6) by the CDM4 LP System which, for the purpose of this test, was arranged so that both data and program resided in core.

The running times reported for the deomposition method, in some instances, exceed those for the direct **LP** solution of the same problem. This disconcerting fact may be explained through consideration of several practical factors. First, we note that our experimental program was written with no regard for programming efficiency, for the sole purpose of solving a limited number of small test cases and investigating some of the computational aspects of the method. Consequently, the timing results should not be compared too closely with those obtained by CDM4, which is an efficient production tool. Second, an appreciable part of our program performs operations which allow the use of CDM4 as a subroutine. Most of this work would not be necessary if a more flexible and versatile linear programming code could be used as a subroutine for the decomposition program. Third, a substantial part of the total solution times consits of the optimization of the sequence of **LM** problems, each one starting from scratch. Comparing Columns 15 and 16, we find that for the test cases treated here, an average of 57% of the total computation time has been expended for solving the **LM** problems. In some cases this percentage exceeds 70%. It is evident, therefore, that considerable savings would result if good starting bases for these **LM** problems were used. In addition, the total computation times for solutions by strategy (b) include the time required for reinversion of the nonoptimal subproblem bases in every cycle. Such reinversions may, of course, be avoided by maintaining the current subproblem inverses in product form, which will result in further savings. The recomputation of the **LM** matrices for optimal blocks, is another expensive operation which may be avoided. Finally, the results reported herein are, in a sense, the worst possible, since solutions to these problems were initiated from an all slack basis and the problem data were generated so that only a small number of these slacks would be contained in the optimal basis. Thus a good starting basis for the complete problem should be expected to improve matters considerably.

Acknowledgments

The authors wish to acknowledge the helpful suggestions of Professor J. B. Rosen, and the contribution to certain sections of the decomposition program by Mr. Dennis Kuba.

References

1. J. M. Abadie and A. C. Williams. Dual and Parametric Methods in Decomposition, "Recent Advances in Mathematical Programming" (P. Wolfe and R. L. Graves, Eds.) pp. 143–158, McGraw-Hill, New York, 1963.
2. E. M. L. Beale. The Simplex Method Using Pseudo-basic Variables for Structured Linear Programming Problems, "Recent Advances in Mathematical Programming" (P. Wolfe, and R. L. Graves, Eds.) pp. 133–148, McGraw-Hill, New York, 1963.
3. J. F. Benders. Partitioning Procedures for Solving Mixed Variables Programming Problems, *Numerische Mathematik* **4**, 238–252 (1962).
4. G. B. Dantzig and P. Wolfe. Decomposition Principle for Linear Programs. *Operations Research* **8**, 101–111 (1961).

5. M. FRANK AND P. WOLFE. An Algorithm for Quadratic Programming, *Naval Res. Logistics Quart.* **3** (1956).
6. A. M. GEOFFRION. Relaxation and the Dual Method in Mathematical Programming, Western Management Science Institute Working Paper No. 135, University of California, Los Angeles (March 1968).
7. M. D. GRIGORIADIS AND K. RITTER. A Decomposition Method for Structured Linear and Nonlinear Programs, Computer Sciences Technical Report #10, University of Wisconsin (January 1968).
8. G. HADLEY. "Linear Programming," Addison-Wesley, Reading, Massachusetts, 1963.
9. P. HUARD. Application du Principe de Décomposition aux Programmes Mathématiques Non-Linéaires, *Note, Electricité de France* HR-547613 (1963).
10. O. L. MANGASARIAN AND J. PONSTEIN. Minmax and duality in Nonlinear Programming, *J. Math. Anal. Appl.* **11**, 504–518 (1965).
11. K. RITTER. A Decomposition Method for Structured Quadratic Programming Problems, *Mathematics Research Center Technical Summary Report* #718, University of Wisconsin (1967).
12. K. RITTER. A Decomposition Method for Linear Programming Problems with Coupling Constraints and Variables, *Mathematics Research Center Technical Summary Report* #739, University of Wisconsin (1967).
13. K. RITTER AND M. D. GRIGORIADIS. A Decomposition Method for Large Structured Non-Linear Programs, *Mathematics Research Center Technical Summary Report* #786, University of Wisconsin (1967).
14. K. RITTER AND M. D. GRIGORIADIS. An Algorithm for Post-Optimality Analysis of Structured Linear Programs, *Mathematics Research Center Technical Summary Report* #815, University of Wisconsin (1967).
15. J. B. ROSEN. The Gradient Projection Method for Non-Linear Programming. Part I. Linear Constraints, *J. Soc. Ind. Appl. Math.*, **8**, 181–217 (1960).
16. J. B. ROSEN. Convex Partition Programming. "Recent Advances in Mathematical Programming" (P. Wolfe and R. L. Graves, Eds.), pp. 159–176, McGraw-Hill, New York, 1963.
17. J. B. ROSEN. Primal Partition Programming for Block Diagonal Matrices, *Numerische Mathematik* **6**, 250–260 (1964).
18. J. B. ROSEN. Parametric Requirement Vector in Primal Partition Programming, Private Communication (1965).
19. J. C. SANDERS. A Non-Linear Decomposition Principle, *Oper. Res.* **13**, 266–271 (1965).
20. D. WEBBER AND W. W. WHITE. A Partitioning Algorithm for Structured Linear Programming Problems, presented at TIMS/OSRA Joint Meeting, San Francisco, May 1968. Also *NY Scientific Center Technical Report* 320–2946, IBM Corporation, April 1968.
21. P. WOLFE. A Duality Theorem in Non-Linear Programming, *Quarterly Appl. Math.*, **XIX**, 239–244 (1961).
22. CDM4: 3400/3600 CDM4-LP System Reference Manual, Control Data Corporation, Palo Alto, California.